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Time-ordered perturbation theory on non-commutative spacetime II: Unitarity

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Abstract. We examine the unitarity issue in the recently proposed time-ordered perturbation theory on noncommutative (NC) spacetime. We show that unitarity is preserved as long as the interaction Lagrangian is explicitly hermitian. We explain why it makes sense to distinguish the hermiticity of the Lagrangian from that of the action in perturbative NC field theory and how this requirement fits in this framework.

1 Introduction

Quantum field theory on non-commutative (NC) spacetime has attracted a lot of attention since it was shown to appear as a limit of string theory in the presence of a constant NS–NS B field background [1]. New features in NC field theory have been found, such as the ultraviolet–infrared mixing [2], violation of unitarity [3,4] and causality [5], which are very alien to ordinary field theory. These results are largely based on the understanding that field theory on NC spacetime may be formulated through the Moyal star product of functions on ordinary spacetime [6] and that the only modification in perturbation theory is the appearance of momentum dependent NC phases at the interaction vertices [7]. This naive approach in perturbation theory has been scrutinized recently in the context of the unitarity problem with the suggestion that the time-ordered product is not properly defined [8]. It has also been shown explicitly that unitarity is preserved for the one-loop two-point function of φ^3 theory in the approach of the Yang–Feldman equation [9].

In a previous work [10], we have reconsidered the issue of NC perturbation theory formulated in terms of the Moyal product. We assumed that perturbation theory can still be developed in the time-ordered expansion of a formally unitary time evolution operator specified by the interaction Lagrangian and that the usual concepts of time-ordering and commutation relations for free fields are still applicable. We found that the result is the old-fashioned, time-ordered perturbation theory (TOPT) [11, 12] naturally extended to the NC case. In this framework, NC phases at the interaction vertices are evaluated at on-shell momenta of positive or negative energy depending on the direction of time flow and are thus independent of the zeroth components of the generally off-shell momenta of participating particles. The analyticity properties of Green functions in the complex plane of the zeroth component are thus significantly modified. We explained how this in turn led to the result that this non-covariant formalism of TOPT cannot be recast into the seemingly covariant form of the naive approach when time does not commute with space. The whole picture of perturbation theory is thus altered; and this difference appears already at tree level in perturbation. It is then quite reasonable to ask whether some of the important statements made in the naive approach will be changed as well. In this work we address the issue of unitarity in the new framework and our conclusion on perturbative unitarity will indeed be different.

In TOPT a process is described as a time sequence of transitions between physical intermediate states. The unitarity property in the TOPT formalism of ordinary field theory is usually transparent. Assuming that NC field theory is renormalizable at higher orders in perturbation theory, the unitarity proof in the TOPT framework for the usual field theory [12] almost goes through without change, up to one caveat which is specific to NC theory. Namely, the interaction Lagrangian must be explicitly hermitian. While the hermiticity of the Lagrangian guarantees that of the action, the opposite is not always automatic in NC field theory. In the naive approach one can appeal to the cyclicity of the spacetime integral of star products for the hermiticity of the action even if one begins with a Lagrangian which is not explicitly hermitian, e.g., $\mathcal{L}_{\rm int} = -g\varphi^\dagger \star \varphi \star \sigma$. However, as we stressed in [10], it is important to notice that the time-ordering procedure does not commute with the star multiplication when time is involved in NC. Strictly insisting on this led to the conclusion that the naive, seemingly covariant approach of NC perturbation theory cannot be recovered from its TOPT formalism. Furthermore, the manip-

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ulation with star products in perturbation theory should not interfere with the time-ordering procedure. And as we shall show below, this has implications for the unitarity problem: to guarantee perturbative unitarity the interaction Lagrangian has to be explicitly hermitian. This means that the above Lagrangian should be replaced by $\mathcal{L}_{\rm int} = -g/2(\varphi^\dagger \star \varphi \star \sigma + \sigma \star \varphi^\dagger \star \varphi)$. While this does no harm to the action and quantities derived from it, it makes a big difference in the fate of unitarity in NC theory.

In the next section we first make an ab initio calculation of the one-loop contribution to the scalar self-energy. The purpose is to show explicitly that the prescriptions given in [10] indeed apply as well to higher orders in perturbation. We confirm its unitarity as required for generally off-shell and amputated Green functions. This is followed by the study of a four-point function arising at one loop using the above prescriptions. Then, we show that it is necessary to make the interaction Lagrangian explicitly hermitian to preserve the perturbative unitarity. This problem already appears at tree level as we shall illustrate by a four-point function. We explain how this requirement fits in the framework of TOPT. We give a summary in the last section.

2 Demonstration of perturbative unitarity

We assume that perturbative field theory on NC spacetime formulated through the Moyal star product of field operators of ordinary spacetime functions can still be developed in terms of vacuum expectation values of time-ordered products of field operators. The basic concepts such as time-ordering and commutation relations for free fields are also assumed to be applicable. This is also the common starting point followed in the literature so far. Of course, there might be a drastic change with these assumptions in NC theory, but the philosophy is that we would like to avoid deviation from the well-established concepts as much as possible. But still, as we showed in the previous work, great differences amongst different approaches arise at a later stage when coping with time-ordered products. In this section, we provide one more difference concerning the fate of unitarity which is important for a theory to be consistent as a quantum theory. As we remarked in the Introduction, unitarity is almost obvious in the framework of TOPT on a formal level; however, an explicit demonstration of this in some examples is still instructive and interesting as we shall present below. Furthermore, it also leads to an observation concerning the hermiticity of the interaction Lagrangian that has been ignored before in the context of NC field theory.

2.1 Self-energy of real scalar field

Let us first study the scalar two-point function that is most frequently discussed in the literature on the unitarity issue. We consider the contribution arising from the following interaction:

$$\mathcal{L}_{\rm int} = -g\left(\chi \star \varphi \star \pi + \pi \star \varphi \star \chi\right), \tag{1}$$

where all fields are real scalars. We have deliberately used three different fields. On the one hand this makes our calculation more general than those considered so far, and on the other hand it avoids unnecessary complications arising from many possible contractions amongst identical fields which may be recovered later on by symmetrization.

The one-loop contribution to the φ two-point function is

$$\begin{aligned} G(x_1,x_2) &= -\frac{g^2}{2!}\int \mathrm{d}^4x_3 \int \mathrm{d}^4x_4 A, \\ A &= \langle 0|T\left(\varphi_1\varphi_2(\chi\star\varphi\star\pi+\pi\star\varphi\star\chi)_3 \right.\\ &\quad \left.\times\ (\chi\star\varphi\star\pi+\pi\star\varphi\star\chi)_4\right)|0\rangle, \end{aligned} \tag{2}$$

where from now on we use the indices of coordinates to specify the fields evaluated at corresponding points when no confusion arises. Our calculation is based on the following commutation relation between the positive- and negative-frequency parts of the field operator, e.g., φ:

$$\begin{aligned} \varphi(x) &= \varphi^+(x) + \varphi^-(x), \\ \left[\varphi^+(x),\varphi^-(y)\right] &= D(x-y) \\ &= \int \mathrm{d}^3\mu_{\boldsymbol{p}} \exp[-\mathrm{i}p_+\cdot(x-y)], \end{aligned} \tag{3}$$

where $\mathrm{d}^3\mu_{\boldsymbol{p}} = \mathrm{d}^3\boldsymbol{p}[(2\pi)^3 2E_{\boldsymbol{p}}]^{-1}$ is the standard phase space measure with $E_{\boldsymbol{p}} = (\boldsymbol{p}^2+m_\varphi^2)^{1/2}$, and $p^\mu_\lambda = (\lambda E_{\boldsymbol{p}}, \boldsymbol{p})$ ($\lambda = \pm$) is the on-shell momentum with positive or negative energy. The calculation proceeds the same way as shown in [10] although new complications arise due to the loop.

We first compute the contractions of the χ and π fields at the interaction points x_3 and x_4. For $x_3^0 > x_4^0$, only χ_4^- and π_4^-, which are on the right, and χ_3^+ and π_3^+, which are on the left, can contribute. Shifting their positions using relations like (3) leads to the result

$$\begin{aligned} A &= +\langle 0|\cdots D_{34}(\chi)\star\varphi_3\cdots\varphi_4\star D_{34}(\pi)\cdots|0\rangle \\ &\quad + \int \mathrm{d}^3\mu_{\boldsymbol{p}}\langle 0|\cdots \mathrm{e}^{-\mathrm{i}p_+^\pi\cdot x_3}\star\varphi_3\star D_{34}(\chi) \\ &\quad \times\cdots\star\varphi_4\star\mathrm{e}^{+\mathrm{i}p_+^\pi\cdot x_4}|0\rangle + (\chi\leftrightarrow\pi), \end{aligned} \tag{4}$$

where $D_{34} = D(x_3-x_4)$ and the argument or index χ (π) refers to the corresponding field and its mass being used. The dots represent possible positions for φ_1 and φ_2 fields appropriate to the time order. The first two terms originate from the diagonal and crossing contractions respectively, while the last arises because the interaction is symmetric in χ and π fields. A little explanation on the star is necessary. Sometimes the same single $\star$ refers to both x_3 and x_4. This only means that the star multiplication is to be done separately with respect to x_3 and x_4. There never arises a case in which a star product is with respect to two different points since the only source of it is the interaction Lagrangian which is defined at a single point. This is a new feature at loop level: that star products at different points get entangled. In principle this is not a problem and should not be confusing when we are careful enough, but it is a problem with notation. For example, the second term in the above equation may also be rewritten in a compact form as the first one; but we find that for this purpose we have to introduce more star symbols and specify which refers to which. This is even worse when more vertices are involved. For our aim of expressing the final result in momentum space, we find this is not worthwhile and it is much better to leave it as it stands. The result for the opposite case of $x_3^0 < x_4^0$ is obtained by interchanging the indices 3 and 4.

Next we contract the φ fields. There are 4! time orders which we certainly do not have to consider one by one. They are classified into six groups: $T_{12}T_{34}$, $T_{34}T_{12}$, $T_{13}T_{24}$, $T_{24}T_{13}$, $T_{14}T_{23}$ and $T_{23}T_{14}$, where $T_{ij}T_{mn}$ stands for $(x_i^0 \text{and} x_j^0) > (x_m^0 \text{and} x_n^0)$. We only need to consider the first three groups while the last three can be obtained from the third by either $3 \leftrightarrow 4$, or $1 \leftrightarrow 2$, or both. The first two groups are relatively easy to compute. For example, for $x_1^0 > x_2^0 > x_3^0 > x_4^0$ which belongs to one of the four possibilities in the first group, we have

$$A = \langle 0|\varphi_1\varphi_2 \cdots \varphi_3 \cdots \varphi_4 \cdots |0\rangle, \tag{5}$$

where the dots now represent the result from χ and π contractions connected by stars which are computed above. There are actually four terms of course. Up to disconnected terms, $\varphi_{1,2}$ may be replaced by $\varphi^+_{1,2}$ and thus $\varphi_{3,4}$ by $\varphi^-_{3,4}$. Pushing further $\varphi^+_{1,2}$ to the right results in the following:

$$\begin{aligned} A = \Big[& D_{34}(\chi) \star (D_{23}D_{14}) \star D_{34}(\pi) \\ & + \int \mathrm{d}^3\mu_{\boldsymbol{p}} \mathrm{e}^{-\mathrm{i}p_+^\pi \cdot x_3} \star (D_{23} \star D_{34}(\chi) \star D_{14}) \star \mathrm{e}^{+\mathrm{i}p_+^\pi \cdot x_4} \\ & + (1 \leftrightarrow 2)\Big] + (\chi \leftrightarrow \pi), \end{aligned} \tag{6}$$

where the D functions without an argument refer to the φ field. The above is symmetric in $x_{1,2}$ and thus applies to the case of $(x_1^0 \text{and} x_2^0) > x_3^0 > x_4^0$. In this way we obtain the results for the first two groups of time orders,

$$\begin{aligned} A = \tau_{34}\tau_{13}\tau_{23} \Big\{ \Big[& D_{34}(\chi) \star (D_{23}D_{14}) \star D_{34}(\pi) \\ & + \int \mathrm{d}^3\mu_{\boldsymbol{p}} \mathrm{e}^{-\mathrm{i}p_+^\pi \cdot x_3} \star (D_{23} \star D_{34}(\chi) \star D_{14}) \star \mathrm{e}^{+\mathrm{i}p_+^\pi \cdot x_4} \\ & + (1 \leftrightarrow 2)\Big] + (\chi \leftrightarrow \pi) \Big\} + (3 \leftrightarrow 4), \quad \text{for} T_{12}T_{34}, \end{aligned} \tag{7}$$

$$\begin{aligned} A = \tau_{34}\tau_{41}\tau_{42} \Big\{ \Big[& D_{34}(\chi) \star (D_{32}D_{41}) \star D_{34}(\pi) \\ & + \int \mathrm{d}^3\mu_{\boldsymbol{p}} \mathrm{e}^{-\mathrm{i}p_+^\pi \cdot x_3} \star (D_{32} \star D_{34}(\chi) \star D_{41}) \star \mathrm{e}^{+\mathrm{i}p_+^\pi \cdot x_4} \\ & + (1 \leftrightarrow 2)\Big] + (\chi \leftrightarrow \pi) \Big\} + (3 \leftrightarrow 4), \quad \text{for} T_{34}T_{12}, \end{aligned} \tag{8}$$

where $\tau_{jk} = \tau(x_j^0 - x_k^0)$ is the step function.

The contractions for the case $T_{13}T_{24}$ is more complicated. Consider one of the four orders, $x_1^0 > x_3^0 > x_2^0 > x_4^0$, for which we have the following structure:

$$A = \langle 0| \cdots \varphi_1\varphi_3 \cdots \varphi_2\varphi_4 \cdots |0\rangle. \tag{9}$$

Note that there is no problem for $\varphi_{1,2}$ to pass over the dots which contain the star products of c-number functions with respect to $x_{3,4}$. Then, $\varphi_1\varphi_3$ may be replaced by $(\varphi_1^+\varphi_3^+ + D_{13})$ and $\varphi_2\varphi_4$ by $(\varphi_2^-\varphi_4^- + D_{24})$. Up to disconnected terms, the remaining product of operators contributes a term $D_{32} \cdots D_{14}$ so that,

$$\begin{aligned} A = & D_{34}(\chi) \star (D_{13}D_{24} + D_{32}D_{14}) \star D_{34}(\pi) \\ & + \int \mathrm{d}^3\mu_{\boldsymbol{p}} \mathrm{e}^{-\mathrm{i}p_+^\pi \cdot x_3} \star (D_{13} \star D_{34}(\chi) \star D_{24} \\ & + D_{32} \star D_{34}(\chi) \star D_{14}) \star \mathrm{e}^{+\mathrm{i}p_+^\pi \cdot x_4} + (\chi \leftrightarrow \pi). \end{aligned} \tag{10}$$

The complete sum for the case $T_{13}T_{24}$ can be put in a compact form,

$$\begin{aligned} A = & (\tau_{1324} + \tau_{1342} + \tau_{3124} + \tau_{3142}) \\ & \times \Big\{ D_{34}(\chi) \star (D_{32}D_{14}) \star D_{34}(\pi) \\ & + \int \mathrm{d}^3\mu_{\boldsymbol{p}} \mathrm{e}^{-\mathrm{i}p_+^\pi \cdot x_3} \star (D_{32} \star D_{34}(\chi) \star D_{14}) \star \mathrm{e}^{+\mathrm{i}p_+^\pi \cdot x_4} \Big\} \\ & + \sum_{\lambda,\lambda'} \tau_{13}^\lambda \tau_{(13)(24)} \tau_{24}^{\lambda'} \Big\{ D_{34}(\chi) \star (D_{13}^\lambda D_{24}^{\lambda'}) \star D_{34}(\pi) \\ & + \int \mathrm{d}^3\mu_{\boldsymbol{p}} \mathrm{e}^{-\mathrm{i}p_+^\pi \cdot x_3} \star (D_{13}^\lambda \star D_{34}(\chi) \star D_{24}^{\lambda'}) \star \mathrm{e}^{+\mathrm{i}p_+^\pi \cdot x_4} \Big\} \\ & + (\chi \leftrightarrow \pi), \end{aligned} \tag{11}$$

where $\tau_{ijmn} = \tau_{ij}\tau_{jm}\tau_{mn}$, $\tau_{(ij)(mn)} = \tau(\min(x_i^0, x_j^0) - \max(x_m^0, x_n^0))$, and

$$\tau_{jk}^\lambda = \begin{cases} \tau_{jk}, \\ \tau_{kj}, \end{cases} \quad D_{jk}^\lambda = \begin{cases} D_{jk}, \text{for} \lambda = +, \\ D_{kj}, \text{for} \lambda = -. \end{cases} \tag{12}$$

Now we show how to sum over 16 pairs of terms (not counting $\chi \leftrightarrow \pi$) so obtained in a desired form; namely, the connected contribution contains only functions $D_{13}^\pm$, $D_{14}^\pm$, $D_{23}^\pm$, $D_{24}^\pm$ and $D_{34}^\pm$ which should be accompanied by the corresponding step functions. We found four of them are already in the desired form. There are eight pairs, each of which is a combination of contributions from two time orders; for example, $\tau_{13}\tau_{23}\tau_{34}$ and $\tau_{13}\tau_{32}\tau_{24}$ unify precisely into the desired one $\tau_{13}\tau_{24}\tau_{34}$. Each of the remaining four pairs is again a combination of two contributions with one of them being of the same type as the first term in (11). They also unify comfortably into the desired time order; for example, the time order in the first term of (11) unifies with the one, $\tau_{32}\tau_{21}\tau_{14}$ from the other contribution, into $\tau_{14}\tau_{34}\tau_{32}$. Therefore, we have finally,

$$A = \sum_{\lambda_1,\lambda_2,\lambda} \Big\{ \tau_{13}^{\lambda_1}\tau_{24}^{\lambda_2}\tau_{34}^{\lambda} \left[D_{34}^\lambda(\chi) \star (D_{13}^{\lambda_1} D_{24}^{\lambda_2}) \star D_{34}^\lambda(\pi) \right]$$

$$
\begin{aligned}
&+ (3 \leftrightarrow 4)\Big\} \\
&+ \sum_{\lambda_1,\lambda_2} \Big\{ \tau_{13}^{\lambda_1}\tau_{24}^{\lambda_2} \int \mathrm{d}^3\mu_{\boldsymbol{p}} \\
&\times \Big[\mathrm{e}^{-\mathrm{i}p_+^\pi\cdot x_3} \star (D_{13}^{\lambda_1} \star D_{34}(\chi) \star D_{24}^{\lambda_2}) \star \mathrm{e}^{+\mathrm{i}p_+^\pi\cdot x_4} \\
&+ \mathrm{e}^{-\mathrm{i}p_+^\pi\cdot x_4} \star (D_{24}^{\lambda_2} \star D_{43}(\chi) \star D_{13}^{\lambda_1}) \star \mathrm{e}^{+\mathrm{i}p_+^\pi\cdot x_3} \Big] \\
&+ (3 \leftrightarrow 4)\Big\} + (\chi \leftrightarrow \pi). \qquad (13)
\end{aligned}
$$

Upon integrating over $x_{3,4}$, $(3 \leftrightarrow 4)$ gives a factor of 2 to cancel 1/2! in (2) from the perturbation series, as expected. The trick to proceed further is the same as employed in [10]. Using

$$
\begin{aligned}
\tau_{jk}^{\lambda} &= \frac{\mathrm{i}\lambda}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}s \frac{\exp[-\mathrm{i}s(x_j^0 - x_k^0)]}{s + \mathrm{i}\epsilon\lambda}, \\
D_{jk}^{\lambda} &= \int \mathrm{d}^3\mu_{\boldsymbol{p}} \exp[-\mathrm{i}p_\lambda\cdot(x_j - x_k)], \qquad (14)
\end{aligned}
$$

we can combine the τ function and its related 3-momentum integral into a 4-momentum integral. To make the result more symmetric in internal χ and π lines, we may replace τ_{34}^{λ} by its square. We checked that the result is identical to the one using one factor of τ_{34}^{λ} as it must be as in ordinary field theory. We skip the further details and write down the result directly:

$$
\begin{aligned}
&G(x_1, x_2) \\
&= -g^2 \int \frac{\mathrm{d}^4p}{(2\pi)^4}\int \frac{\mathrm{d}^4q}{(2\pi)^4}\int \frac{\mathrm{d}^4p_1}{(2\pi)^4}\int \frac{\mathrm{d}^4p_2}{(2\pi)^4} \\
&\times \sum_{\lambda_1,\lambda_2,\lambda} \Big\{ \mathrm{i}P_\lambda(p)\mathrm{i}P_\lambda(q)\mathrm{i}P_{\lambda_1}(p_1)\mathrm{i}P_{\lambda_2}(p_2)\mathrm{e}^{-\mathrm{i}p_1\cdot x_1}\mathrm{e}^{-\mathrm{i}p_2\cdot x_2} \\
&\times (2\pi)^4\delta^4(p_1 + p_2)(2\pi)^4\delta^4(p + q - p_1) \\
&\times (\text{NC vertices})\Big\}, \qquad (15)
\end{aligned}
$$

where $p, q, p_{1,2}$ refer to the π, χ, φ fields respectively so that their masses are implicit in on-shell quantities such as $E_{\boldsymbol{p}}$ and p_λ, and

$$
\begin{aligned}
P_\lambda(k) &= \frac{\lambda}{2E_{\boldsymbol{k}}[k^0 - \lambda(E_{\boldsymbol{k}} - \mathrm{i}\epsilon)]} \\
&= \frac{\eta_\lambda(k)}{k^2 - m^2 + \mathrm{i}\epsilon}, \qquad (16)
\end{aligned}
$$

with $\eta_\lambda(k) = 1/2(1 + \lambda k_0/E_{\boldsymbol{k}})$. The vertices have the factorized form,

$$
\begin{aligned}
&(\text{NC vertices}) \\
&= \Big[\mathrm{e}^{-\mathrm{i}(q_\lambda,-p_{1\lambda_1},p_\lambda)} + \mathrm{e}^{-\mathrm{i}(p_\lambda,-p_{1\lambda_1},q_\lambda)}\Big] \\
&\times \Big[\mathrm{e}^{-\mathrm{i}(q_\lambda,+p_{2\lambda_2},p_\lambda)} + \mathrm{e}^{-\mathrm{i}(p_\lambda,+p_{2\lambda_2},q_\lambda)}\Big], \qquad (17)
\end{aligned}
$$

with $(k_1, k_2, \cdots, k_n) = \sum_{i<j} k_i \wedge k_j$ and $p \wedge q = 1/2\theta_{\mu\nu}p^\mu q^\nu$.

Transforming into momentum space is now straightforward,

$$
\begin{aligned}
\hat{G}(k_1, k_2) &= \prod_{j=1}^{2}\Big[\int \mathrm{d}^4x_j \mathrm{e}^{-\mathrm{i}k_j\cdot x_j}\Big] G(x_1, x_2) \\
&= -g^2(2\pi)^4\delta^4(k_1 + k_2)\sum_{\lambda_1,\lambda_2} \mathrm{i}P_{\lambda_1}(k_1)\mathrm{i}P_{\lambda_2}(k_2) \\
&\times \sum_\lambda \int \frac{\mathrm{d}^4p}{(2\pi)^4}\int \frac{\mathrm{d}^4q}{(2\pi)^4}(2\pi)^4\delta^4(p + q - k_1) \\
&\times \mathrm{i}P_\lambda(p)\mathrm{i}P_\lambda(q)(\text{NC vertices}), \qquad (18)
\end{aligned}
$$

where $k_{1,2}$ are the momenta flowing into the diagram. We have reversed the signs of variables λ_j, λ and p, q so that the only change in NC vertices is, $p_{1\lambda_1} \to k_{1\lambda_1}, p_{2\lambda_2} \to k_{2\lambda_2}$. As we pointed out in [10], it is important to notice that the zeroth components p_0 and q_0 are not involved in NC vertices which contain only on-shell momenta of positive or negative energy. This fact changes the analyticity properties significantly. The p_0, q_0 integrals can thus be finished, one by the δ function, the other by a contour in its complex plane, with the result,

$$
\begin{aligned}
&\mathrm{i}^{-1}\hat{G}(k_1, k_2)\mathrm{i}^{-1}(k_1^2 - m_\varphi^2)\mathrm{i}^{-1}(k_2^2 - m_\varphi^2) \\
&= -g^2(2\pi)^4\delta^4(k_1 + k_2)\sum_{\lambda_1,\lambda_2}\eta_{\lambda_1}(k_1)\eta_{\lambda_2}(k_2) \\
&\times \sum_\lambda \int \mathrm{d}^3\mu_{\boldsymbol{p}} \int \mathrm{d}^3\mu_{\boldsymbol{q}}(2\pi)^3\delta^3(\boldsymbol{p} + \boldsymbol{q} - \boldsymbol{k}_1) \\
&\times \frac{(\text{NC vertices})}{\lambda k_1^0 - E_{\boldsymbol{p}} - E_{\boldsymbol{q}} + \mathrm{i}\epsilon}, \qquad (19)
\end{aligned}
$$

which is exactly the amputated two-point function as can be obtained directly from the prescriptions given in [10].

We are now ready to examine the unitarity problem. We found that unitarity holds true in a detailed sense. Namely, as in ordinary field theory, it holds not only for the on-shell transition matrix but also for the off-shell amputated Green function. In the current non-covariant formalism, it even holds for separate configurations of external time direction parameters λ_j. This is not surprising since, if kinematically allowed, we can obtain the S-matrix elements for all possible channels of physical processes from the same Green function, which just correspond to different configurations of λ_j and satisfy the unitarity relation. Let us check the following unitarity relation for the above example. Assuming $T(\{k_i, \lambda_i\} \to \{k_f, \lambda_f\})$ is the transition matrix or the amputated Green function for the process $i \to f$ with incoming momenta and time parameters $\{k_i, \lambda_i\}$ and outgoing ones $\{k_f, \lambda_f\}$, we have

$$
\begin{aligned}
&-\mathrm{i}\left[T(\{k_i, \lambda_i\} \to \{k_f, \lambda_f\}) - T^*(\{k_f, \lambda_f\} \to \{k_i, \lambda_i\})\right] \\
&= \sum_n \prod_{j=1}^{n}\Big[\int \mathrm{d}^3\mu_{\boldsymbol{p}_j}\Big] \\
&\times T(\{k_i, \lambda_i\} \to n)T^*(\{k_f, \lambda_f\} \to n), \qquad (20)
\end{aligned}
$$

where n is a physical intermediate state with n particles.

Following the above convention, we change in our example $k_2 \to -k_2$ and $\lambda_2 \to -\lambda_2$ so that $k_{2\lambda_2} \to -k_{2\lambda_2}$, and the transition matrix becomes

$$\begin{aligned} T(\{k_1,\lambda_1\} \to \{k_2,\lambda_2\}) &= -g^2(2\pi)^4\delta^4(k_1-k_2) \\ &\times \sum_\lambda \int \mathrm{d}^3\mu_{\boldsymbol{p}} \int \mathrm{d}^3\mu_{\boldsymbol{q}} (2\pi)^3\delta^3(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{k}_1) \\ &\times \frac{V(1\to n)V(2\to n)}{\lambda k_1^0 - E_{\boldsymbol{p}} - E_{\boldsymbol{q}} + \mathrm{i}\epsilon}, \end{aligned} \quad (21)$$

with $V(j \to n) = 2\cos(q_\lambda, -k_{j\lambda_j}, p_\lambda)$ being real so that only the physical threshold can develop an imaginary part,

$$\begin{aligned} &-\mathrm{i}\left[T(\{k_1,\lambda_1\} \to \{k_2,\lambda_2\}) - T^*(\{k_2,\lambda_2\} \to \{k_1,\lambda_1\})\right] \\ &= +g^2(2\pi)^4\delta^4(k_1-k_2) \\ &\times \sum_\lambda \int \mathrm{d}^3\mu_{\boldsymbol{p}} \int \mathrm{d}^3\mu_{\boldsymbol{q}} (2\pi)^3\delta^3(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{k}_1) \\ &\times 2\pi\delta(\lambda k_1^0 - E_{\boldsymbol{p}} - E_{\boldsymbol{q}}) V(1\to n)V(2\to n). \end{aligned} \quad (22)$$

For a given sign of k_1^0, only one term in the sum over λ actually contributes while the sum automatically includes both cases. For $k_1^0 < 0$, it is physically better to work with the inverse process. The above is precisely what we obtain for the right-hand side of (20) using the prescriptions for the two transitions; it is, for example, for $k_1^0 > 0$,

$$\begin{aligned} &\int \mathrm{d}^3\mu_{\boldsymbol{p}} \int \mathrm{d}^3\mu_{\boldsymbol{q}} \Big[(2\pi)^3\delta^3(\boldsymbol{k}_1-\boldsymbol{p}-\boldsymbol{q}) \\ &\times (-2\pi)\delta(k_1^0 - E_{\boldsymbol{p}} - E_{\boldsymbol{q}}) gV(1\to n)\Big] \\ &\times \Big[(2\pi)^3\delta^3(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{k}_2)(-2\pi)\delta(E_{\boldsymbol{p}}+E_{\boldsymbol{q}}-k_2^0) \\ &\times gV(2\to n)\Big]^*, \end{aligned} \quad (23)$$

and unitarity is thus verified. For the complete and amputated Green function we merely have to multiply a real factor of $\eta_{\lambda_j}(k_j)$ for each external line and sum over λ_j. The on-shell transition matrix is projected by $k_j^0 \to \lambda_j E_{\boldsymbol{k}_j}$. These manipulations do not lead to further problems. We note that the reality of the NC vertices originating from the hermiticity of $\mathcal{L}_{\text{int}}$ in (1) plays an important role. Coping with a single real scalar field would not make this point so clear since $\mathcal{L}_{\text{int}}$ would be automatically hermitian. The latter case may be recovered by symmetrization which is already clear from our previous study.

2.2 Four-point function of real scalar field

As a second example to demonstrate unitarity and to show applications of the prescriptions in [10], we consider the φ four-point function arising from the interaction,

$$\mathcal{L}_{\text{int}} = -g\varphi \star (\chi\star\pi + \pi\star\chi)\star\varphi, \quad (24)$$

where all fields are real again. The lowest order contribution arises at one loop which has three Feynman diagrams. Here we shall consider only one of them, shown in

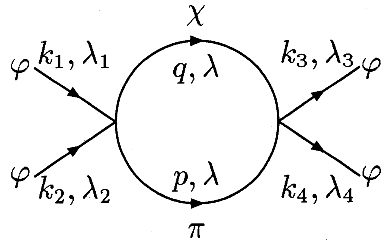

Fig. 1. Diagram corresponding to (25)

Fig. 1, while the other two may be obtained by permutation of indices. Their unitarity may be checked separately. Each Feynman diagram corresponds to two time-ordered diagrams which are represented collectively in Fig. 1 by the parameter $\lambda = \pm$. For the configuration of incoming $k_{1,2}, \lambda_{1,2}$ and outgoing $k_{3,4}, \lambda_{3,4}$, we have

$$\begin{aligned} T(12\to 34) &= -2\pi\delta(k_1^0+k_2^0-k_3^0-k_4^0)\int \mathrm{d}^3\mu_{\boldsymbol{p}}\int \mathrm{d}^3\mu_{\boldsymbol{q}} \\ &\times \sum_\lambda \left[(2\pi)^3\delta^3(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{p}-\boldsymbol{q})gV_{12}\right] \\ &\times \left[(2\pi)^3\delta^3(\boldsymbol{k}_3+\boldsymbol{k}_4-\boldsymbol{p}-\boldsymbol{q})gV_{34}\right] \\ &\times \left[\lambda(k_1^0+k_2^0) - E_{\boldsymbol{p}} - E_{\boldsymbol{q}} + \mathrm{i}\epsilon\right]^{-1}, \end{aligned} \quad (25)$$

where

$$\begin{aligned} V_{12} &= \left[\mathrm{e}^{-\mathrm{i}(k_{1\lambda_1}, -q_\lambda, -p_\lambda, k_{2\lambda_2})} + (q_\lambda \leftrightarrow p_\lambda)\right] \\ &\quad + (k_{1\lambda_1} \leftrightarrow k_{2\lambda_2}), \\ V_{34} &= \left[\mathrm{e}^{-\mathrm{i}(-k_{3\lambda_3}, q_\lambda, p_\lambda, -k_{4\lambda_4})} + (q_\lambda \leftrightarrow p_\lambda)\right] \\ &\quad + (k_{3\lambda_3} \leftrightarrow k_{4\lambda_4}). \end{aligned} \quad (26)$$

$(q_\lambda \leftrightarrow p_\lambda)$ is due to the hermitian arrangement of the χ and π fields in $\mathcal{L}_{\text{int}}$, while $(k_{1\lambda_1} \leftrightarrow k_{2\lambda_2})$ or $(k_{3\lambda_3} \leftrightarrow k_{4\lambda_4})$ is from symmetrization in the two φ fields. We thus have

$$\begin{aligned} V_{12} &= 2^2\cos[k_{1\lambda_1}\wedge k_{2\lambda_2} + (p_\lambda+q_\lambda)\wedge(k_{1\lambda_1}-k_{2\lambda_2})] \\ &\quad \times \cos(p_\lambda \wedge q_\lambda), \\ V_{34} &= 2^2\cos[k_{3\lambda_3}\wedge k_{4\lambda_4} + (p_\lambda+q_\lambda)\wedge(k_{3\lambda_3}-k_{4\lambda_4})] \\ &\quad \times \cos(p_\lambda \wedge q_\lambda), \end{aligned} \quad (27)$$

which are real again and do not contribute to the imaginary part of the transition matrix. For the φ^4 interaction of a single real scalar field, one merely has to symmetrize V_{12} and V_{34} further by including all permutations.

The left-hand side of the unitarity relation is

$$\begin{aligned} &-\mathrm{i}[T(12\to 34) - T^*(34\to 12)] \\ &= (-\mathrm{i})(-2\pi)(-\mathrm{i}2\pi)\delta(k_1^0+k_2^0-k_3^0-k_4^0) \\ &\times \sum_\lambda \int \mathrm{d}^3\mu_{\boldsymbol{p}}\int \mathrm{d}^3\mu_{\boldsymbol{q}} \left[(2\pi)^3\delta^3(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{p}-\boldsymbol{q})gV_{12}\right] \\ &\times \left[(2\pi)^3\delta^3(\boldsymbol{k}_3+\boldsymbol{k}_4-\boldsymbol{p}-\boldsymbol{q})gV_{34}\right] \\ &\times \delta\left(\lambda(k_1^0+k_2^0) - E_{\boldsymbol{p}} - E_{\boldsymbol{q}}\right), \end{aligned} \quad (28)$$

which becomes, e.g., for $k_1^0 + k_2^0 > 0$,

$$\int \mathrm{d}^3\mu_{\boldsymbol{p}}\int \mathrm{d}^3\mu_{\boldsymbol{q}} \left[-(2\pi)^4\delta^4(k_1+k_2-p_+-q_+)gV_{12}\right]$$

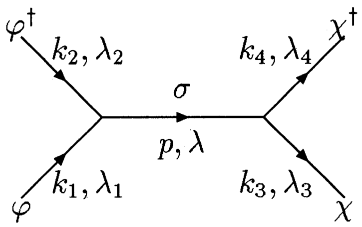

Fig. 2. Diagram corresponding to (32)

$$\times \left[-(2\pi)^4\delta^4(p_+ + q_+ - k_3 - k_4) g V_{34}\right], \qquad (29)$$

precisely the right-hand side of the unitarity relation. We have also checked that the above result is identical to that of an ab initio calculation.

2.3 Hermiticity of Lagrangian and unitarity of S-matrix

From the above two examples it is already clear that the hermiticity of the interaction Lagrangian is crucial to preserving unitarity as it is in ordinary theory. But they also seem to indicate that only real NC couplings are allowed for this purpose. If this were the case, NC field theory would be much less interesting as far as the standard model is concerned. We would like to clarify this point in this subsection; namely, there is no obstacle for a complex NC coupling to appear and the only requirement is the explicit hermiticity of the interaction Lagrangian. We also point out the difference in the hermiticity of the Lagrangian and action that is specific to perturbative NC field theory and that has not been noticed so far. In the meanwhile, we shall explain how this difference fits in the framework of TOPT. All of these points cannot be properly realized in φ^3 or φ^4 theory of a single real scalar, that has been most frequently used in the literature in this context.

Let us consider the following interaction Lagrangian:

$$\mathcal{L}'_{\rm int} = -g_\varphi \varphi^\dagger \star \varphi \star \sigma - g_\chi \chi^\dagger \star \chi \star \sigma, \qquad (30)$$

where σ (φ, χ) is a real (complex) scalar and the coupling constants $g_{\varphi,\chi}$ are real. According to the understanding in the naive approach, the above is well defined in the sense that the action $S'_{\rm int} = \int {\rm d}^4x \mathcal{L}'_{\rm int}$ is hermitian by using the cyclicity property of integrals of star products, although $\mathcal{L}'_{\rm int}$ is not in itself. The cyclicity argument in turn is based on integration by parts and ignoring surface terms. However, this integration by parts, when involving time derivatives, may clash with the time-ordering procedure in perturbation theory expanded in $S'_{\rm int}$. Thus the above argument may break down in perturbation theory and cause problems. We have seen in [10] a similar case of non-commutativity, i.e. that the time-ordering procedure does not commute with star multiplication making the naive approach not recoverable from the TOPT framework when time does not commute with space.

To see the unitarity problem resulting from (30), it is sufficient to consider the transition matrix for the following scattering at tree level:

$$\varphi(k_1,\lambda_1) + \varphi^\dagger(k_2,\lambda_2) \to \chi(k_3,\lambda_3) + \chi^\dagger(k_4,\lambda_4), \quad (31)$$

where $(k_{1,2}, \lambda_{1,2})$ are incoming while $(k_{3,4}, \lambda_{3,4})$ are outgoing. k_j's are not necessarily on-shell, and λ_j's are meaningful only when their connections to vertices are specified as we do in Fig. 2. The T-matrix for a fixed configuration of λ_j's is,

$$\begin{aligned} T(12 \to 34) &= (-2\pi)\delta(k_1^0 + k_2^0 - k_3^0 - k_4^0) \int {\rm d}^3\mu_{\boldsymbol{p}} \\ &\times \sum_\lambda \left[(2\pi)^3\delta^3(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{p}) g_\varphi V'_{12}\right] \\ &\times \left[(2\pi)^3\delta^3(\boldsymbol{p} - \boldsymbol{k}_3 - \boldsymbol{k}_4) g_\chi V'_{34}\right] \\ &\times \left[\lambda(k_1^0 + k_2^0) - E_{\boldsymbol{p}} + {\rm i}\epsilon\right]^{-1} \\ &= -(2\pi)^4\delta^4(k_1 + k_2 - k_3 - k_4) \\ &\times \sum_\lambda \frac{g_\varphi V'_{12} g_\chi V'_{34}}{2E_{\boldsymbol{p}}[\lambda(k_1^0 + k_2^0) - E_{\boldsymbol{p}} + {\rm i}\epsilon]}, \end{aligned} \qquad (32)$$

where $\boldsymbol{p} = \boldsymbol{k}_1 + \boldsymbol{k}_2$, $E_{\boldsymbol{p}} = (\boldsymbol{p}^2 + m_\sigma^2)^{1/2}$ and similarly for other energies. The NC vertices are,

$$\begin{aligned} V'_{12} &= \exp[-{\rm i}(k_{2\lambda_2}, k_{1\lambda_1}, -p_\lambda)], \\ V'_{34} &= \exp[-{\rm i}(k_{3\lambda_3}, k_{4\lambda_4}, -p_\lambda)], \end{aligned} \qquad (33)$$

where we have used $(-a,-b,c) = (a,b,-c)$ for V'_{34}. For the inverse transition of incoming $(k_{3,4}, \lambda_{3,4})$ and outgoing $(k_{1,2}, \lambda_{1,2})$, we have,

$$\begin{aligned} T(34 \to 12) &= (-2\pi)\delta(k_3^0 + k_4^0 - k_1^0 - k_2^0) \int {\rm d}^3\mu_{\boldsymbol{p}} \\ &\times \sum_\lambda \left[(2\pi)^3\delta^3(\boldsymbol{k}_3 + \boldsymbol{k}_4 - \boldsymbol{p}) g_\chi \bar{V}'_{34}\right] \\ &\times \left[(2\pi)^3\delta^3(\boldsymbol{p} - \boldsymbol{k}_1 - \boldsymbol{k}_2) g_\varphi \bar{V}'_{12}\right] \\ &\times \left[\lambda(k_3^0 + k_4^0) - E_{\boldsymbol{p}} + {\rm i}\epsilon\right]^{-1} \\ &= -(2\pi)^4\delta^4(k_1 + k_2 - k_3 - k_4) \\ &\times \sum_\lambda \frac{g_\varphi \bar{V}'_{12} g_\chi \bar{V}'_{34}}{2E_{\boldsymbol{p}}[\lambda(k_1^0 + k_2^0) - E_{\boldsymbol{p}} + {\rm i}\epsilon]}, \end{aligned} \qquad (34)$$

with

$$\begin{aligned} \bar{V}'_{12} &= \exp[-{\rm i}(k_{1\lambda_1}, k_{2\lambda_2}, -p_\lambda)], \\ \bar{V}'_{34} &= \exp[-{\rm i}(k_{4\lambda_4}, k_{3\lambda_3}, -p_\lambda)]. \end{aligned} \qquad (35)$$

Noting that $V'_{ij} \neq \bar{V}'^*_{ij}$; these factors will not factorize when forming the difference on the left-hand side of the unitarity relation, $-{\rm i}\,[T(12 \to 34) - T^*(34 \to 12)]$. On the other hand, the right-hand side of the relation factorizes of course,

$$\begin{aligned} &\sum_\lambda \int {\rm d}^3\mu_{\boldsymbol{p}} \\ &\times \left[-(2\pi)\delta(k_1^0 + k_2^0 - \lambda E_{\boldsymbol{p}})(2\pi)^3\delta^3(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{p}) g_\varphi V'_{12}\right] \\ &\times \left[-(2\pi)\delta(k_3^0 + k_4^0 - \lambda E_{\boldsymbol{p}})(2\pi)^3\delta^3(\boldsymbol{k}_3 + \boldsymbol{k}_4 - \boldsymbol{p}) g_\chi \bar{V}'^*_{34}\right] \\ &= (2\pi)^4\delta^4(k_1 + k_2 - k_3 - k_4)\frac{1}{2E_{\boldsymbol{p}}} \\ &\times \sum_\lambda 2\pi\delta(k_1^0 + k_2^0 - \lambda E_{\boldsymbol{p}}) g_\varphi V'_{12} g_\chi \bar{V}'^*_{34}. \end{aligned} \qquad (36)$$

Unitarity is thus violated at tree level.

Let us take a closer look at what goes wrong in the above. For example, using the spatial momentum conservation at the vertex, we have,

$$2(k_{1\lambda_1}, k_{2\lambda_2}, -p_\lambda) = \theta_{0i}\Big[(\lambda E_{\boldsymbol{p}} - \lambda_1 E_{\boldsymbol{k}_1} - \lambda_2 E_{\boldsymbol{k}_2})p^i + (\lambda_2 E_{\boldsymbol{k}_2} k_1^i - \lambda_1 E_{\boldsymbol{k}_1} k_2^i)\Big] + \theta_{ij} k_1^i k_2^j. \quad (37)$$

Requiring $V'_{jk} = \bar{V}'^{*}_{jk}$ ($jk = 12, 34$), which guarantees unitarity, amounts to vanishing of the following:

$$(k_{1\lambda_1}, k_{2\lambda_2}, -p_\lambda) + (k_{2\lambda_2}, k_{1\lambda_1}, -p_\lambda) = \theta_{0i}(\lambda E_{\boldsymbol{p}} - \lambda_1 E_{\boldsymbol{k}_1} - \lambda_2 E_{\boldsymbol{k}_2})p^i, \quad (38)$$

and similarly for $jk = 34$. We make a few observations on the above result.

First, space-space NC does not pose a problem with unitarity in the framework of TOPT. Namely, even if one starts with a Lagrangian such as (30) which can only be made hermitian by the cyclicity property, there will be no problem as long as $\theta_{0i} = 0$. This is because the time-ordering procedure in perturbation theory does not interfere with the partial integration of spatial integrals employed in the cyclicity property. Furthermore, this freedom in partial integration corresponds exactly to the spatial momentum conservation at each separate vertex of TOPT. Conversely, we do not have such a freedom in temporal integration, which would spoil the time-ordering procedure, so that the temporal component of momentum does not conserve at each separate vertex in TOPT. But still we have a global conservation law for it which corresponds to the same amount of shift for all time parameters without disturbing their relative order.

Second, for the particular example considered here, when all external particles are on-shell, we still have a chance to saturate unitarity even if $\theta_{0i} \neq 0$. For instance, when all $k_j^0 = E_{\boldsymbol{k}_j}$ and $\lambda_j = +$, only $\lambda = +$ contributes to the unitarity relation so that unitarity holds true if $E_{\boldsymbol{p}} = E_{\boldsymbol{k}_1} + E_{\boldsymbol{k}_2} = E_{\boldsymbol{k}_3} + E_{\boldsymbol{k}_4}$. For the off-shell transition, which is a sum over all configurations of λ_j and λ, (38) cannot always vanish and thus there is no unitarity for the off-shell function. For transitions involving more than one internal line or loops so that we may have more freedom in spatial momenta of intermediate states, vanishing of similar combinations cannot be generally fulfilled. Thus we should not rely on this even for a cure to S-matrix unitarity. The same comment also applies to the kinematical configuration of $\theta_{0i}p^i = 0$.

The solution to this problem is already clear from the above discussion. Whenever time-space NC enters, we should make the interaction Lagrangian explicitly hermitian before we do perturbation. In our example, instead of (30), we should start with the following one:

$$\mathcal{L}_{\rm int} = -\frac{g_\varphi}{2}(\varphi^\dagger \star \varphi \star \sigma + \sigma \star \varphi^\dagger \star \varphi) - \frac{g_\chi}{2}(\chi^\dagger \star \chi \star \sigma + \sigma \star \chi^\dagger \star \chi). \quad (39)$$

The only effect of this rearrangement is the substitution of the above primed vertices by the following ones:

$$V_{12} = \frac{1}{2}\left[\mathrm{e}^{-\mathrm{i}(k_{2\lambda_2}, k_{1\lambda_1}, -p_\lambda)} + \mathrm{e}^{-\mathrm{i}(-p_\lambda, k_{2\lambda_2}, k_{1\lambda_1})}\right],$$
$$V_{34} = \frac{1}{2}\left[\mathrm{e}^{-\mathrm{i}(k_{3\lambda_3}, k_{4\lambda_4}, -p_\lambda)} + \mathrm{e}^{-\mathrm{i}(-p_\lambda, k_{3\lambda_3}, k_{4\lambda_4})}\right], \quad (40)$$

and then

$$\bar{V}_{12} = V_{12}^*, \quad \bar{V}_{34} = V_{34}^*, \quad (41)$$

which guarantees unitarity for any configurations since the difference on the left-hand side of the unitarity relation arises only from the physical threshold. More explicitly, we have, e.g.,

$$V_{12} = \exp(-\mathrm{i}k_{2\lambda_2} \wedge k_{1\lambda_1}) \cos\left((k_{1\lambda_1} + k_{2\lambda_2}) \wedge p_\lambda\right), \quad (42)$$

which is a complex coupling indeed.

As a final example, we would like to illustrate the interplay between the hermiticity of the Lagrangian and the contributions from complex conjugate intermediate states. We consider the one-loop induced $\sigma \to \rho$ transition through the following interactions:

$$\mathcal{L}_{\rm int} = -(g_\sigma \chi^\dagger \star \pi \star \sigma + g_\sigma^* \sigma \star \pi^\dagger \star \chi) - (g_\rho \chi \star \pi^\dagger \star \rho + g_\rho^* \rho \star \pi \star \chi^\dagger), \quad (43)$$

where χ, π are complex scalars and ρ, σ are real ones with generally complex couplings g_σ, g_ρ. There are two Feynman diagrams with conjugate virtual particle pairs $\chi\pi^\dagger$ and $\chi^\dagger\pi$ respectively, and each of them has two time-ordered diagrams depicted collectively in Fig. 3. We write down their contributions directly,

$$T_{\chi\pi^\dagger}(\sigma(k_1,\lambda_1) \to \rho(k_2,\lambda_2)) = -(2\pi)^4\delta^4(k_1 - k_2) g_\sigma g_\rho \times \int \mathrm{d}^3\mu_{\boldsymbol{p}} \int \mathrm{d}^3\mu_{\boldsymbol{q}} (2\pi)^3 \delta^3(\boldsymbol{k}_1 - \boldsymbol{p} - \boldsymbol{q}) \times \sum_\lambda \frac{V_\sigma V_\rho}{\lambda k_1^0 - E_{\boldsymbol{p}} - E_{\boldsymbol{q}} + \mathrm{i}\epsilon},$$
$$T_{\chi^\dagger\pi}(\sigma(k_1,\lambda_1) \to \rho(k_2,\lambda_2)) = -(2\pi)^4\delta^4(k_1 - k_2) g_\sigma^* g_\rho^* \times \int \mathrm{d}^3\mu_{\boldsymbol{p}} \int \mathrm{d}^3\mu_{\boldsymbol{q}} (2\pi)^3 \delta^3(\boldsymbol{k}_1 - \boldsymbol{p} - \boldsymbol{q}) \times \sum_\lambda \frac{V_\sigma^* V_\rho^*}{\lambda k_1^0 - E_{\boldsymbol{p}} - E_{\boldsymbol{q}} + \mathrm{i}\epsilon}, \quad (44)$$

with $V_\sigma = \mathrm{e}^{-\mathrm{i}(q_\lambda, p_\lambda, -k_{1\lambda_1})}$, $V_\rho = \mathrm{e}^{-\mathrm{i}(q_\lambda, p_\lambda, -k_{2\lambda_2})}$. Similarly, for the inverse transition $\rho \to \sigma$, we have

$$T_{\chi\pi^\dagger}(\rho(k_2,\lambda_2) \to \sigma(k_1,\lambda_1)) = -(2\pi)^4\delta^4(k_1 - k_2) g_\sigma^* g_\rho^* \times \int \mathrm{d}^3\mu_{\boldsymbol{p}} \int \mathrm{d}^3\mu_{\boldsymbol{q}} (2\pi)^3 \delta^3(\boldsymbol{k}_1 - \boldsymbol{p} - \boldsymbol{q}) \times \sum_\lambda \frac{V_\sigma^* V_\rho^*}{\lambda k_1^0 - E_{\boldsymbol{p}} - E_{\boldsymbol{q}} + \mathrm{i}\epsilon},$$

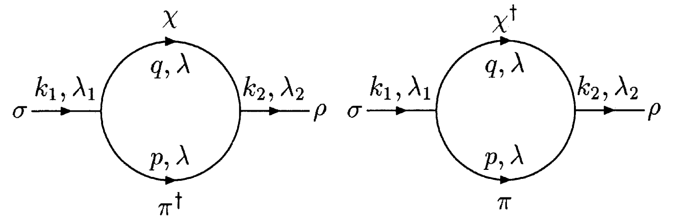

Fig. 3. Diagrams corresponding to (44)

$$
\begin{aligned}
& T_{\chi^\dagger \pi}(\rho(k_2, \lambda_2) \to \sigma(k_1, \lambda_1)) \\
& = -(2\pi)^4 \delta^4(k_1 - k_2) g_\sigma g_\rho \\
& \times \int \mathrm{d}^3\mu_{\boldsymbol{p}} \int \mathrm{d}^3\mu_{\boldsymbol{q}} (2\pi)^3 \delta^3(\boldsymbol{k}_1 - \boldsymbol{p} - \boldsymbol{q}) \\
& \times \sum_\lambda \frac{V_\sigma V_\rho}{\lambda k_1^0 - E_{\boldsymbol{p}} - E_{\boldsymbol{q}} + \mathrm{i}\epsilon}.
\end{aligned} \tag{45}
$$

Thus, with g_σ and g_ρ coupling terms alone (or their conjugate terms alone) it is impossible to fulfill unitarity because even the action is not hermitian. If the action is hermitian, it is always possible to make the Lagrangian hermitian too. Once this is done, unitarity holds individually for the two conjugate intermediate states of $\chi\pi^\dagger$ and $\chi^\dagger\pi$. This is precisely the same phenomenon occurring in ordinary field theory as may be easily checked for the above example.

3 Conclusion

In a previous paper we proposed a framework to do perturbation theory for NC field theory which is essentially the time-ordered perturbation theory extended to the NC case. In contrast to ordinary field theory, this framework is not equivalent to the naive, seemingly covariant one pursued in the literature due to the significant change of analyticity properties introduced by NC phases. In the present paper we examined the impact of this change on the unitarity problem occurring in the naive approach when time does not commute with space, and arrived at a different result on the fate of unitarity. Our main conclusion is that there is no problem with unitarity in TOPT as long as the interaction Lagrangian is explicitly hermitian. We showed this explicitly in examples and then extended that result.

The key observation in distinguishing the hermiticity of the Lagrangian and that of the action in NC field theory is that the manipulation with the cyclicity property in spacetime integrals of star products may clash with the time-ordering procedure in perturbation theory. In ordinary theory there is no similar problem arising from integration by parts in the action because, even if one takes it seriously at the beginning, one can always remove it by going back to the covariant formalism by analytic continuation. However, in NC theory with time-space NC, as we argued previously, this continuation is not possible at least in the naive sense. It thus makes difference whether the Lagrangian is explicitly hermitian or not. But we would like to stress again that requiring hermiticity of the Lagrangian does not forbid complex NC couplings to appear.

The main drawback of TOPT, as it is in ordinary theory, is its rapidly increased technical complication when going to higher orders in perturbation. It would be highly desirable if it could be recast in a more or less covariant form.

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References

1. A. Connes, M.R. Douglas, A. Schwarz, J. High Energy Phys. **02**, 003 (1998) [hep-th/9711162]; M.R. Douglas, C. Hull, ibid. **02**, 008 (1998) [hep-th/9711165]; C.-S. Chu, P.-M. Ho, Nucl. Phys. B **550**, 151 (1999) [hep-th/9812219]; ibid. B **568**, 447 (2000) [hep-th/9906192]; V. Schomerus, J. High Energy Phys. **06**, 030 (1999) [hep-th/9903205]; N. Seiberg, E. Witten, ibid. **09**, 032 (1999) [hep-th/9908142]
2. S. Minwalla, M.V. Raamsdonk, N. Seiberg, J. High Energy Phys. **02**, 020 (2000) [hep-th/9912072]; I. Ya. Aref'eva, D.M. Belov, A.S. Koshelev, Phys. Lett. B **476**, 431 (2000) [hep-th/9912075]; M.V. Raamsdonk, N. Seiberg, J. High Energy Phys. **03**, 035 (2000) [hep-th/0002186]; A. Matusis, L. Susskind, N. Toumbas, J. High Energy Phys. **12**, 002 (2000) [hep-th/0002075]
3. J. Gomis, T. Mehen, Nucl. Phys. B **591**, 265 (2000) [hep-th/0005129]
4. For subsequent discussions on unitarity, see for example: J. Gomis, K. Kamimura, J. Llosa, Phys. Rev. D **63**, 045003 (2001) [hep-th/0006235]; O. Aharony, J. Gomis, T. Mehen, J. High Energy Phys. **09**, 023 (2000) [hep-th/0006236]; M. Chaichian, A. Demichev, P. Presnajder, A. Tureanu, Eur. Phys. J. C **20**, 767 (2001) [0007156]; R.-G. Cai, N. Ohta, J. High Energy Phys. **10**, 036 (2000) [hep-th/0008119]; L. Alvarez-Gaume, J.L.F. Barbon, R. Zwicky, J. High Energy Phys. **05**, 057 (2001) [hep-th/0103069]; T. Mateos, A. Moreno, Phys. Rev. D **64**, 047703 (2001) [hep-th/0104167]; A. Bassetto, L. Griguolo, G. Nardelli, F. Vian, J. High Energy Phys. **07**, 008 (2001) [hep-th/0105257]; C.-S. Chu, J. Lukierski, W.J. Zakrzewski, Hermitian analyticity, IR/UV mixing and unitarity of non-commutative field theories, hep-th/0201144
5. N. Seiberg, L. Susskind, N. Toumbas, J. High Energy Phys. **06**, 044 (2000) [hep-th/0005015]; L. Alvarez-Gaume, J.L.F. Barbon, Int. J. Mod. Phys. A **16**, 1123 (2001) [hep-th/0006209]
6. S. Doplicher, K. Fredenhagen, J.E. Roberts, Commun. Math. Phys. **172**, 187 (1995)
7. T. Filk, Phys. Lett. B **376**, 53 (1996)
8. D. Bahns, S. Doplicher, K. Fredenhagen, G. Piacitelli, Phys. Lett. B **533**, 178 (2002) [hep-th/0201222]; see also: C. Rim, J.H. Yee, hep-th/0205193
9. C.N. Yang, D. Feldman, Phys. Rev. **79**, 972 (1950)
10. Y. Liao, K. Sibold, Time-ordered perturbation theory on non-commutative spacetime: basic rules, hep-th/0205269
11. S.S. Schweber, An introduction to relativistic quantum field theory (Harper & Row, 1961)
12. G. Sterman, An introduction to quantum field theory (Cambridge University Press, 1993)